RELATIVE KNOT INVARIANTS: PROPERTIES AND APPLICATIONS

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Abstract. We state Bennequin inequalities in the relative case, and show that the relative invariants are additive under relative connected sums. We show they exhibit similar limitations as their classical analogues. We study relatively Legendrian simple knots and give some classification results.

1. Introduction

Classifying Legendrian and transverse knots in contact 3-manifolds has been an important part of the recent development of 3-manifold topology. One of the breakthroughs in this direction came about with the work of GIroux and the theory of convex surfaces (see [1, 10, 11, 13]). Ideas of convex surface theory are usually applied to null-homologous knots in a contact 3-manifold. Our goal is to apply them in the case when a knot is homologous to another "reference" knot.

In [17], we defined the following relative invariants.

Definition 1.1. Let K and J be homologous Legendrian knots in a contact 3-manifold (M,ξ) oriented accordingly with $K\cup J=\partial\Sigma$ for an oriented embedded Seifert surface Σ so that $[\partial\Sigma]=[K]-[J]$. Define the *Thurston-Bennequin invariant of K relative to J* by

$$\widetilde{tb}_\Sigma(K,J):=tw_K(\xi,Fr_\Sigma)-tw_J(\xi,Fr_\Sigma),$$

where Fr_Σ denotes the Seifert framing that K (resp. J) inherits from Σ, and $tw(\xi,Fr_\Sigma)$ denotes the number of 2π-twists (with sign) of the contact framing relative to Fr_Σ along K or J. For push-offs K' and J' of K and J in the direction normal to the contact planes, $\widetilde{tb}_\Sigma(K,J)=K'\cdot\Sigma-J'\cdot\Sigma=lk_\Sigma(K',K)-lk_\Sigma(J',J)$.

Definition 1.2. Let K and J be homologous Legendrian knots in a contact 3-manifold (M,ξ) oriented accordingly with $K\cup J=\partial\Sigma$ for an oriented embedded Seifert surface Σ so that $[\partial\Sigma]=[K]-[J]$. The restriction to K of the trivialized contact 2-plane field $\xi|_\Sigma$ gives a map $\sigma:\xi|_K\to K\times\mathbb{R}^2$, under which a non-zero tangent vector field v_K to K traces out a path of vectors in $\mathbb{R}^2$. We can then compute the winding number $w_\sigma(v_K)$ and similarly for J. Then define the *relative rotation number of K* by

$$\widetilde{r}_\Sigma(K,J):=w_\sigma(v_K)-w_\sigma(v_J).$$

Equivalently, $\widetilde{r}_\Sigma(K,J)=e(\xi,v_K\cup v_J)([\Sigma])$.

Definition 1.3. Let K and J be homologous transverse knots in a contact 3-manifold (M,ξ) oriented accordingly with $K\cup J=\partial\Sigma$ for an oriented embedded Seifert surface Σ so that $[\partial\Sigma]=[K]-[J]$. The contact 2-plane field ξ is trivial over

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Σ, so there exists a nonzero vector field v in $\xi|_\Sigma$. Take K' and J' to be the push offs of K and J in the direction of v. Then define the *relative self-linking number* of K with respect to J by

$$\widetilde{sl}_\Sigma(K, J) := K' \cdot \Sigma - J' \cdot \Sigma.$$

In what follow, we establish relative versions of the Bennequin inequalities and develop some prototypical examples. We describe relative connected sums of Legendrian and transverse knots and study the additivity of the relative invariants, following the foundational work of Etnyre-Honda [11]. We show that the relative invariants exhibit similar limitations as their classical analogues, in particular, the relative Thurston-Bennequin invariant and the relative rotation number are not able to distinguish relative connected sums of the Chekanov knots [3] which are smoothly isotopic, have equal relative invariants, but are not Legendrian isotopic. We study basic knot types which can be classified by their relative invariants, and give a generalization of the structure theorem of Etnyre-Honda [11] which classifies Legendrian knots in a relative knot type in terms of their relative connected sum prime components.

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3. Background

We briefly recall some facts from contact geometry and convex surface theory. This is far from a complete introduction to the subject, and the reader should consult the more complete treatment in [1, 10, 11, 13].

Definition 3.1. An (transversely) oriented positive contact structure ξ on M is an oriented 2-plane field $\xi \subset TM$ for which there is a 1-form α such that $\xi = \ker \alpha$ and $\alpha \wedge d\alpha > 0$ (recall that M is oriented).

Two contact structures ξ_1, ξ_2 on a 3-manifold M are *homotopic* if they are homotopic as 2-plane distributions. They are *isotopic* if they are homotopic through contact structures. They are *contactomorphic* if there is a diffeomorphism $f : M \to M$ such that f sends one of the contact structures to the other, i.e., $f_*(\xi_1) = \xi_2$. Then f is called a *contactomorphism*.

Perturbing a contact structure occurs only through perturbing the ambient manifold, as the theorem below states.

Theorem 3.2 (Gray Stability)**.** *Given a 1-parameter family of contact structures $\xi_t, t \in [0, 1]$, there is a 1-parameter family of diffeomorphisms $f_t : M \to M$ such that $(f_t)_*(\xi_0) = \xi_t$ for all t.*

A smooth oriented embedding of S^1 in a contact 3-manifold (M, ξ) is called a *Legendrian knot* if it is everywhere tangent to the contact planes. It is a *transverse knot* if it is everywhere transverse to the contact planes.

If $K \subset \Sigma$ is a simple closed Legendrian curve in an embedded surface Σ in a contact 3-manifold (M, ξ), then $tw_\Sigma(K)$ is the *twisting* of ξ along K relative to

the Seifert framing Fr_Σ. That is, both ξ and Σ give K a framing (a trivialization of its normal bundle) by taking a vector field normal to K and tangent to ξ or Σ, respectively (note that ξ is trivializable over Σ). Then $tw_\Sigma(K)$ measures the number of 2π-twists (as we traverse the oriented K) of the vector field corresponding to ξ relative to the vector field coming from Σ. By convention, left-handed twists are negative and right-handed twists are positive. Equivalently, take a push-off K' of K along a vector field transverse to ξ. Then $tw_\Sigma(K)$ is equal the signed intersection of K' with Σ, $tw_\Sigma(K) = K' \cdot \Sigma$, or the linking number of K with K'.

Let Σ be an oriented compact surface embedded in a contact 3-manifold (M, ξ). If $\partial\Sigma$ is nonempty, assume that it is Legendrian. Then the line field $l_p = \xi_p \cap T_p\Sigma$, $p \in \Sigma$, integrates to a singular foliation on Σ called the *characteristic foliation*, denoted Σ_ξ.

The contact structure ξ is called *overtwisted* if there is an embedded disc D such that D_ξ contains a closed leaf. Such a disc is called an overtwisted disc. If there are no overtwisted discs in ξ, then the contact structure is called *tight*.

Now we turn to the theory of convex surfaces, which have been a very useful tool in the study of 3-dimensional contact manifolds.

Definition 3.3. Let Σ be an oriented compact surface embedded in a contact 3-manifold (M, ξ). If $\partial\Sigma$ is nonempty, assume that it is Legendrian. Then Σ is called *convex* if there exists a *contact vector field* X that is transverse to Σ (a contact vector field X is a vector field whose flow preserves the contact structure).

Any closed surface is C^∞-close to a convex surface. If Σ has Legendrian boundary with $tw_\Sigma(K) \leq 0$ for all components K of $\partial\Sigma$, then after a C^0-small perturbation of Σ near the boundary (but fixing the boundary), Σ will be C^∞-close to a convex surface.

Definition 3.4. Let Σ be a convex surface with X a transverse contact vector field. The set $\Gamma_\Sigma = \{p \in \Sigma \mid X_p \in \xi_p\}$ is an embedded multi-curve on Σ called the *dividing* set.

Proposition 3.5. *Let $\mathcal{F}$ be a singular foliation on Σ and let Γ be a multi-curve on Sigma. The multi-curve is said to* divide $\mathcal{F}$ *if*

(a) Γ_Σ *is transverse to* $\mathcal{F}$
(b) $\Sigma \setminus \Gamma_\Sigma = \Sigma_+ \amalg \Sigma_-$
(c) there is a vector field X and a volume form ω on Σ such that
  (i) X directs $\mathcal{F}$ (that is, it is tangent to $\mathcal{F}$ at non-singular points and $X = 0$ at the singular points of $\mathcal{F}$)
  (ii) the flow of X expands ω on Σ_+ and contracts ω on Σ_-
  (iii) and X points transversely out of Σ_+.

Theorem 3.6 (Giroux's Criterion)**.** *Let Σ be a convex surface in a contact 3-manifold (M, ξ). Then Σ has a tight neighborhood in M if and only if $\Sigma \neq S^2$ and Γ_Σ contains no contractible curves or $\Sigma = S^2$ and Γ_Σ is connected.*

4. Generalized Bennequin inequalities

Let Σ be an embedded surface in a contact 3-manifold (M, ξ) with $\partial\Sigma \neq \emptyset$ having multiple components. Let $\mathfrak{F}$ be the singular characteristic foliation on Σ. Isotop Σ (C^∞-small) away from $\partial\Sigma$ so that the singularities of $\mathfrak{F}$ are isolated elliptic and hyperbolic (see [8, 13]). Let $e_\pm$ be the number of positive/negative elliptic

singularities and $h_\pm$ be the number of positive/negative hyperbolic singularities. The Poincaré-Hopf theorem says that $\chi(\Sigma) = (e_+ + e_-) - (h_+ + h_-)$.

For a transverse knot K with Seifert surface Σ in a contact 3-manifold (M, ξ), consider a non-zero section in the trivialization $\xi|_\Sigma$ and take a push-off K' of K along this section. The *self-linking number of* K is defined by $sl(K) := K' \cdot \Sigma$. Let $e(\xi)([\Sigma])$ denote the Euler class of $\xi|_\Sigma$. Let $\mathfrak{F}$ be the characteristic foliation which flows transversely out of $K = \partial\Sigma$. Consider the graph $G = \{(x, p) \in \xi|_\Sigma \mid p = v(x), x \in \Sigma, p \in \xi_x\}$ of v which directs $\mathfrak{F}$. G is a surface in the 4-manifold $\xi|_\Sigma$, and the zero section is another surface given by $\{(x, 0) \in \xi|_\Sigma \mid x \in \Sigma\}$, so the Euler class of $\xi|_\Sigma$ is the oriented intersection number of these two surfaces. Counting singularities with signs, we have $-sl(K) = e(\xi)([\Sigma]) = (e_+ - h_+) - (e_- - h_-)$. The Poincaré-Hopf theorem yields Eliashberg's equation $\chi(\Sigma) + e(\xi)([\Sigma]) = 2(e_+ - h_+)$. Convex surface theory gives us that $e_+ = 0$ (see [6, 8]) and we obtain the classical Bennequin inequality $sl(K) \leq -\chi(\Sigma)$ ([2]).

This approach generalizes directly for Σ with transverse $\partial\Sigma = K_1 \cup \cdots \cup K_m$, we have $-(sl(K_1) + \cdots + sl(K_m)) = e(\xi)([\Sigma]) = (e_+ - h_+) - (e_- - h_-)$.

Lemma 4.1. *(Generalized Bennequin inequality) Given Σ with transverse $\partial\Sigma = K_1 \cup \cdots \cup K_m$ in a tight contact 3-manifold, $sl(K_1) + \cdots + sl(K_m) \leq -\chi(\Sigma)$.*

Following Eliashberg [6] and Etnyre [8, 9], consider a Legendrian knot K with Seifert surface Σ and an annulus $A = S^1 \times [-1, 1]$ in a standard neighborhood around K such that A is transverse to ξ and K is the only closed leaf on the characteristic foliation of A. Then take the union of Σ with the appropriate part of A to form a Seifert surface $\Sigma_\pm$ for the knot $\gamma_\pm = S^1 \times \{\pm 1\}$. If the neighborhood is chosen so that ∂A, the $\Sigma_\pm$ are isotopic to Σ, and the Euler characteristic of $\chi(\Sigma_\pm) = \chi(\Sigma)$ because the part of A in each Seifert surface does not contribute to $\chi(\Sigma_\pm)$. Then $sl(K_{1\pm}) + \cdots + sl(K_{m\pm}) = \Big(tw_{K_1}(\xi, Fr_\Sigma) \mp r(K_1)\Big) + \cdots + \Big(tw_{K_m}(\xi, Fr_\Sigma) \mp r(K_m)\Big)$.

Lemma 4.2. *(Generalized Thurston-Bennequin inequality) Given Σ with transverse $\partial\Sigma = K_1 \cup \cdots \cup K_m$ in a tight contact 3-manifold, we have*

$$tw_{K_1}(\xi, Fr_\Sigma) + \cdots + tw_{K_m}(\xi, Fr_\Sigma) + |r(K_1) + \cdots + r_(K_m)| \leq -\chi(\Sigma).$$

This observation has several important consequences.

Lemma 4.3. *Let K and J be homologous Legendrian knots in a tight contact 3-manifold (M, ξ), then $\widetilde{tb}_\Sigma(K, J)$ is bounded above.*

Proof. By Lemma 4.2, $tw_K(\xi, Fr_\Sigma) + tw_J(\xi, Fr_\Sigma) + |r(K) + r(J)| \leq -\chi(\Sigma)$ yields $\widetilde{tb}_\Sigma(K, J) + |r_\Sigma(K) + r_\Sigma(J)| \leq -\chi(\Sigma) - 2tw_J(\xi, Fr_\Sigma)$ or $\widetilde{tb}_\Sigma(K, J) \leq -\chi(\Sigma) - 2tw_J(\xi, Fr_\Sigma)$. The quantity $-\chi(\Sigma) - 2tw_J(\xi, Fr_\Sigma)$ is fixed because J is fixed. $\square$

This argument generalizes directly for a knot homologous to multiple knots.

Lemma 4.4. *If $K, J_1, \ldots, J_m$ are Legendrian with $K \cup J_1 \cup J_2 \cup \cdots \cup J_m = \partial\Sigma$ in a tight contact 3-manifold (M, ξ), then $\widetilde{tb}_\Sigma(K, J_1 \cup \cdots \cup J_m)$ is bounded above.*

Remark 4.5. The above bound depends on the J_i while $\widetilde{tb}_n(K, J)$ in Theorem 6.7 is bounded by $Tb(\varphi(K)) - n$ for $\varphi(K) \subset (S^3, \xi_{std})$ and even though J is also fixed, instead of using the Seifert surface for $K \cup J$ directly, we find a Seifert surface Σ' for $\varphi(K)$ and use the bound on $tb_{\Sigma'}(\varphi(K))$. We want to compare the two approaches. Since $\Sigma' = \Sigma \cup D$, $\chi(\Sigma') = \chi(\Sigma) + \chi(D)$. The two bounds are

$\widetilde{tb}_\Sigma(K,J) \leq Tb_{\Sigma'}(\varphi(K)) - tw_J(\xi_n, Fr_\Sigma)$ and $\widetilde{tb}_\Sigma(K,J) \leq -\chi(\Sigma) - 2tw_J(\xi_n, Fr_\Sigma)$. Since $Tb_{\Sigma'}(\varphi(K)) \leq -\chi(\Sigma')$, we have $\widetilde{tb}_\Sigma(K,J) \leq -\chi(\Sigma') - tw_J(\xi_n, Fr_\Sigma)$. Also $tb_D(\varphi(K)) + |r_D(\varphi(K))| \leq -\chi(D)$ and $tb_D(K_0) = tw_J(\xi_n, Fr_\Sigma)$, which implies that $tw_J(\xi_n, Fr_\Sigma) \leq -\chi(D)$, which yields $\widetilde{tb}_\Sigma(K,J) \leq -\chi(\Sigma') - tw_J(\xi_n, Fr_\Sigma)$. Both bounds are smaller than $-\chi(S) - tw_J(\xi_n, Fr_\Sigma)$, but we do not have a direct way of comparing them by just using classical methods. This relates to the problem of the exactness of the Thurston-Bennequin inequality.

5. Additivity of the relative invariants

We study the additivity of the relative invariants under versions of connected sum. The results build up on the work of Etnyre-Honda [11]. Recall the following.

Theorem 5.1. *(Colin [4]) Denote by $Tight(M)$ the space of tight contact 2-plane fields on a 3-manifold M. Then given contact 3-manifolds M_1, M_2, there is an isomorphism $\pi_0(Tight(M_1) \times \pi_0(Tight(M_2)) \xrightarrow{\cong} \pi_0(Tight(M_1 \# M_2))$.*

Remark 5.2. (Contact connected sum [11, 23]) Let (M_i, ξ_i), $i = 1, 2$, be tight contact 3-manifolds. Choose points $p_i \in M_i$ and a *standard contact 3-ball* B_i around each p_i (by Darboux's theorem, $(B_i, \xi_i|_{B_i})$ is contactomorphic to a 3-ball around the origin in (S^3, ξ_{std})). Note ∂B_i is C^∞-close to a convex 2-sphere with a single dividing curve (Giroux's Criterion, [16]), and Giroux's Flexibility Theorem [13] allows us to arrange that ∂B_i have diffeomorphic foliations so the B_i are contactomorphic ([5]), and there is an orientation-reversing diffeomorphism $f : \partial(M_1 \setminus B_1) \to \partial(M_2 \setminus B_2)$ that maps the characteristic foliation on $\partial(M_1 \setminus B_1)$ to the characteristic foliation on $\partial(M_2 \setminus B_2)$. Then the *contact connected sum* $(M_1, \xi_1)\#(M_2, \xi_2) = \big((M_1 \setminus B_1) \cup_f (M_2 \setminus B_2), \xi_1 \#_f \xi_2\big)$ yields a tight contact 3-manifold and is independent of the choice of B_i, p_i, and f. Moreover, every tight contact structure on M arises as the contact connected sum of a unique pair (ξ_1, ξ_2).

Remark 5.3. (Legendrian connected sum [11]) The Legendrian connected sum is a relative version of the contact connected sum. In (S^3, ξ_{std}), it can easily be described using the front projection of two Legendrian knots K_1 and K_2 as joining a right cusp of K_1 and a left cusp of K_2 (well-defined by the uniqueness of the front projection). By Theorem 5.1, the contact structure on $S^3 = (S^3, \xi_{std})\#(S^3, \xi_{std})$ is tight so it is isotopic to ξ_{std}. In the general construction ([11]), pick points $p_i \in K_i \subset M_i$ and neighborhoods B_i of the p_i. Then use an orientation-reversing diffeomorphism $f : \partial B_1 \to \partial B_2$ to construct the contact connected sum $M_1 \# M_2 = (M_1 \setminus B_1) \cup_f (M_2 \setminus B_2)$. This diffeomorphism (Remark 5.2) performs exactly what we observed in the front projection, with the cusps at the points p_i.

Lemma 5.4. *In the connected sum of K_1, K_2, $tb(K_1 \# K_2) = tb(K_1) + tb(K_2) + 1$.*

Proof. This was proved in [11, 23], here we show an argument due to Etnyre (in a personal note) which keeps track of the Seifert surfaces. For Legendrian knots K_1 and K_2 with $K_i = \partial\Sigma_i, i = 1, 2$, pick a small arc a_i on K_i and isotop (the interior of) Σ_i so that there is a positive elliptic singularity on a_i and no other singularities in a small disc $D_i \subset \Sigma_i$ about a_i. Near Σ_i but disjoint from it, pick a disc D'_i with boundary $\partial D'_i = a'_i \cup b_i$ where the arc a'_i has a negative elliptic point, the arc b_i is transverse to ξ_i, and there are no other singularities in D'_i. Now take a Legendrian arc c_i connecting the elliptic points on a_i and a'_i. In $(\mathbb{R}^3, \xi_0)$, take a right cusp in the xz-plane centered on the x-axis lying to the left of the z-axis, a left cusp to the right

of the z-axis, and a Legendrian arc on the x-axis connecting the cusps. There is a contactomorphism of a neighborhood of $D_i \cup c_i \cup D_i'$ to two discs in $(\mathbb{R}^3, \xi_0)$ having the cusps in the boundary and the arc on the y-axis. Now apply the Legendrian connected sum in the front projection to these cusps along the arc on the x-axis. In particular, one can see that the singularity a_i in the characteristic foliation of Σ_i before we perform the connected sum contributes a left-handed half-twist to the twisting of the contact planes along K_i relative to the framing from Σ_i. After the connect sum operation, both these singularities are gone, but all other singularities remain. So there is a net "+1" to the contact plane twisting along the knot relative to the Seifert framing (here, the Seifert surface is given by $(\Sigma_1 \setminus D_1) \cup (\Sigma_2 \setminus D_2)$). □

Lemma 5.5. *(Splitting a Legendrian connected sum) Consider a Legendrian knot $K = \partial\Sigma$ of knot type $\kappa_1 \# \kappa_2$ in a tight contact 3-manifold $(M, \xi) = (M_1, \xi_1) \# (M_2, \xi_2)$. Then K can be split into knots $K_i' \subset (M_i, \xi_i)$ of knot type $\kappa_i \subset (M_i, \xi_i)$ such that $tb(K) = tb(K_1') + tb(K_2') + 1$.*

Proof. We modify the Etnyre-Honda construction in [11] to keep track of the Seifert surfaces. Let $K = \partial\Sigma$ be a Legendrian knot of knot type $\kappa_1 \# \kappa_2$ in a tight contact 3-manifold $(M, \xi) = (M_1, \xi_1) \# (M_2, \xi_2)$. There exists a splitting 2-sphere S for Σ such that $S \cap \Sigma = \alpha$, an arc with $\partial\alpha = \{x_1, x_2\} \subset K$. Isotop Σ so that x_1 is a negative singularity on K and x_2 is a positive singularity on K in the characteristic foliation of Σ (isotop S to make it convex). Note α intersects the dividing set Γ_S in $\{x_1, x_2\}$. Take a closed curve $\gamma \subset S$ containing α and isotop S to Legendrian realize γ (see [19]). Then α is a Legendrian arc on Σ which still intersects $K = \partial\Sigma$ in $\{x_1, x_2\}$. The interior of α contains an odd number of intersections with Γ_S (intuitively, it contains an odd number of half-twists of the contact planes relative to Fr_Σ). Moreover, Γ_S consists of a single closed curve, so the arc $\gamma \subset \Gamma_S$, $\partial\gamma = \{x_1, x_2\}$, which intersects α is "parallel" to α, that is, γ and α co-bound a collection of (an even number of) 2-discs on S. Consider another arc $\alpha' \subset S$ which is parallel to α, is tangent to α at x_1 and x_2, and which contains a single intersection with Γ_S in its interior. Isotop S to Legendrian realize α' and isotop the interior of Σ so that $S \cap \Sigma = \alpha'$. Note that α' contains a single left twist of the contact planes with respect to Fr_S and thus - relative to Fr_Σ. Use α' to complete each of the components of K corresponding to the knot type of K_1 or K_2, respectively. Thus, we obtain two knots $K_i' \subset (M_i, \xi_i)$ of knot type $\kappa_i \subset (M_i, \xi_i)$ with $tb(K) = tb(K_1') + tb(K_2') + 1$. If Σ has any other boundary components, the equality $tb(K) = tb(K_1') + tb(K_2') + 1$ holds in one of its relative versions (see below). □

First we look at a "semi-relative" case when one of the connected summands is homologous to another knot. Note $\widetilde{tb}_\Sigma(K_1 \# K_2, J)$ is well-defined (see [17]).

Proposition 5.6. *Let $K_1, J \subset (M_1, \xi_1)$ be homologous Legendrian knots and $K_2 \subset (M_2, \xi_2)$ be a null-homologous Legendrian knot. Assume the ξ_i are tight, $\widetilde{tb}_{\Sigma_1}(K_1, J) = \widetilde{Tb}(K_1, J)$, and $tb(K_2) = Tb(K_2)$. Then $\widetilde{Tb}(K_1 \# K_2, J) = \widetilde{Tb}(K_1, J) + Tb(K_2) + 1$, where J in $\widetilde{Tb}(K_1 \# K_2, J)$ is the image of $J \subset M_1$ under the connected sum.*

Proof. Because ξ_i are tight, $\widetilde{tb}(K_1, J)$ and $tb(K_2)$ are bounded. For Legendrian knots K_1, K_2, J as above, Lemma 5.4 gives $\widetilde{Tb}(K_1, J) + Tb(K_2) + 1 = \widetilde{tb}(K_1 \# K_2, J) \leq \widetilde{Tb}(K_1 \# K_2, J)$. Conversely, if $\widetilde{tb}(K_1 \# K_2, J) = \widetilde{Tb}(K_1 \# K_2, J)$, then Lemma 5.5 implies that $\widetilde{Tb}(K_1 \# K_2, J) = \widetilde{tb}(K_1, J) + tb(K_2) + 1 \leq \widetilde{Tb}(K_1, J) + Tb(K_2) + 1$. □

The above also follows from the relative structure theorem (Theorem 12.1) and directly extends to the case when both summands are homologous to another knot.

Proposition 5.7. *For homologous Legendrian knots* $K_i, J_i \subset (M_i, \xi_i)$ *with* $K_i \cup J_i = \partial S_i, i = 1, 2$. *Assume* ξ_i *is tight, and* $\tilde{tb}(K_1 \# K_2, J_1 \cup J_2) = \widetilde{Tb}(K_1 \# K_2, J_1 \cup J_2)$. *Then* $\widetilde{Tb}(K_1 \# K_2, J_1 \cup J_2) = \widetilde{Tb}(K_1, J) + \widetilde{Tb}(K_2, J_2) + 1$, *where* $J_1 \cup J_2$ *in the term* $\widetilde{Tb}(K_1 \# K_2, J_1 \cup J_2)$ *is a knot in* $M_1 \# M_2$.

Remark 5.8. (Relative Legendrian connected sum) Consider homologous Legendrian knots $K_i, J_i \subset (M_i, \xi_i)$ with Seifert surface $\Sigma_i, i = 1, 2$. Take an arc $\alpha_i \subset \Sigma_i$ with $\partial \alpha_i \subset \partial \Sigma_i$ such that α_i runs from K_i to J_i.

Take a neighborhood $B_i \subset M_i$ of α_i (with convex boundary) and an orientation-reversing diffeomorphism $f : \partial B_1 \to \partial B_2$. Form the connected sum $M_1 \#_{\alpha_1, \alpha_2} M_2 = (M_1 \setminus B_1) \cup_f (M_2 \setminus B_2)$ as follows. First, $B_i \cap \Sigma_i$ is a 2-disc $D_i \subset \Sigma_i$ with four corners, a_i, b_i, c_i, d_i, whose boundary ∂D_i is a union of four arcs. Isotop the interior of Σ so that the four corners of ∂D_i are singularities in the foliation of Σ.

We have $\partial D_i = \gamma_{K_i} \cup \gamma_{J_i} \cup \gamma_i' \cup \gamma_i''$ where $\gamma_{K_i} \subset K_i$ with $\partial \gamma_{K_i} = \{a_i, b_i\}$, $\gamma_{J_i} \subset J_i$ with $\partial \gamma_{J_i} = \{c_i, d_i\}$ (Figure 14). Isotop ∂B_i to Legendrian realize ∂D_i and isotop the interior of D_i to make it convex. By tightness, $tb(D_i) \leq -1$. Also, γ_{K_i} (resp., γ_{J_i}) intersects Γ_{D_i} once and contains a negative half-twist along K_i (resp., J_i).

Now take an orientation-reversing diffeomorphism $f : (M_1 \setminus B_1) \to (M_2 \setminus B_2)$ such that $f(\partial D_1) = \partial D_2$ and $f(a_1) = a_2,\ f(b_1) = b_2,\ f(c_1) = c_2,\ f(d_1) = d_2$, so that $f(\gamma_{K_1})$ is isotopic to γ_{K_2}, $f(\gamma_{J_1})$ is isotopic to γ_{K_2}, $f(\gamma_1')$ is isotopic to γ_2', and $f(\gamma_1'')$ is isotopic to γ_2'' (rel boundary as unoriented arcs). Then in $M_1 \#_{\alpha_1, \alpha_2} M_2$, the knots $K = (K_1 \setminus \gamma_{K_1}) \cup_f (K_2 \setminus \gamma_{K_2})$ and $J = (J_1 \setminus \gamma_{J_1}) \cup_f (J_2 \setminus \gamma_{J_2})$ co-bound a surface $\Sigma = (\Sigma_1 \setminus D_1) \cup_f (\Sigma_2 \setminus D_2)$ and are of type $K_1 \#_{\alpha_1, \alpha_2} K_2$ and $J_1 \#_{\alpha_1, \alpha_2} J_2$, respectively. Moreover, by construction $tw_K(\xi, Fr_\Sigma) = tw_{K_1}(\xi_1, Fr_{\Sigma_1}) + tw_{K_2}(\xi_2, Fr_{\Sigma_2}) + 1$ and $tw_J(\xi, Fr_\Sigma) = tw_{J_1}(\xi_1, Fr_{\Sigma_1}) + tw_{J_2}(\xi_2, Fr_{\Sigma_2}) + 1$.

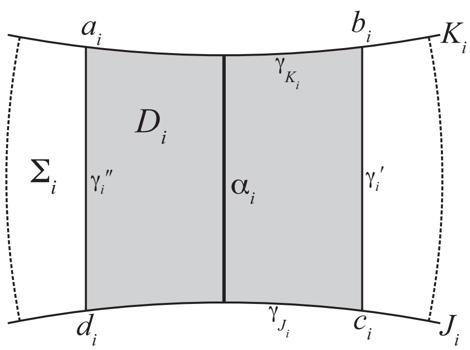

FIGURE 1. Local setup for a relative Legendrian connected sum.

The diffeomorphism type of $\Sigma_{\alpha_1, \alpha_2}$ with $\partial \Sigma_{\alpha_1, \alpha_2} = (K_1 \#_{\alpha_1, \alpha_2} K_2) \cup (J_1 \#_{\alpha_1, \alpha_2} J_2)$ and the link type of $K_1 \#_{\alpha_1, \alpha_2} K_2 \cup J_1 \#_{\alpha_1, \alpha_2} J_2$ depend on the choice of the arcs α_i so this construction is not well-defined as a connected sum of the links $K_i \cup J_i$.

Lemma 5.9. *Consider homologous Legendrian knots* K_i, J_i *in a tight contact 3-manifold* $(M_i, \xi_i), i = 1, 2$. *In* $M_1 \#_{\alpha_1, \alpha_2} M_2 = (M_1 \setminus B_1) \cup_f (M_2 \setminus B_2)$, *the*

diffeomorphism type of $M_1\#_{\alpha_1,\alpha_2}M_2$, the isotopy type of $\xi_1\#_{\alpha_1,\alpha_2}\xi_2$, and the knot type of $K_1\#_{\alpha_1,\alpha_2}K_2$ and $J_1\#_{\alpha_1,\alpha_2}J_2$ are independent of the choices of the α_i, B_i, f.

Proof. Consider the relative Legendrian connected sums of (K_1, J_1, M_1, ξ_1) with (K_2, J_2, M_2, ξ_2) along two sets of arcs $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$. Then $M_1\#_{\alpha_1,\alpha_2}M_2$ and $M_1\#_{\beta_1,\beta_2}M_2$ are diffeomorphic as smooth manifolds because any two 3-balls in a connected 3-manifold are isotopic, which extends to a global isotopy between the diffeomorphisms f_α and f_β. Moreover, Colin's theorem gives that $\xi_1\#_{\alpha_1,\alpha_2}\xi_2$ and $\xi_1\#_{\beta_1,\beta_2}\xi_2$ are isotopic. The knots $K_1\#_{\alpha_1,\alpha_2}K_2$ and $K_1\#_{\beta_1,\beta_2}K_2$ are of the same knot type in $M_1\#M_2$ (γ_{K_i} do depend on the choice of α_i or β_i but we can isotop the B_i (slide them along K_i and J_i) so that γ_{K_i} coincide for either α_i or β_i). Similarly, $J_1\#_{\alpha_1,\alpha_2}J_2$ and $J_1\#_{\beta_1,\beta_2}J_2$ are of the same knot type in $M_1\#M_2$. □

Lemma 5.10. *In the relative Legendrian connected sum of (K_1, J_1, M_1, ξ_1) and (K_2, J_2, M_2, ξ_2), $tw_{K_1\#K_2}(\xi_1\#\xi_2, Fr_{\Sigma_1\#\Sigma_2})$ and $tw_{J_1\#J_2}(\xi_1\#\xi_2, Fr_{\Sigma_1\#\Sigma_2})$ are well-defined and independent of the choice of arcs α_i.*

Proof. Consider the relative Legendrian connected sum of (K_1, J_1, M_1, ξ_1) and (K_2, J_2, M_2, ξ_2) along two sets of arcs α_1, α_2 and β_1, β_2, where $\alpha_i, \beta_i \subset \Sigma_i$. Then we can isotop α_i so that $\partial\alpha_i$ coincides with $\partial\beta_i$, thus the arcs γ_{K_i} and γ_{J_i} are the same for both α_i and β_i. Therefore, under the diffeomorphism between $M_1\#_{\alpha_i,\alpha_2}M_2$ and $M_1\#_{\beta_i,\beta_2}M_2$, $K_1\#_{\alpha_1,\alpha_2}K_2$ is sent to $K_1\#_{\beta_1,\beta_2}K_2$ and $J_1\#_{\alpha_1,\alpha_2}J_2$ is sent to $J_1\#_{\beta_1,\beta_2}J_2$. This diffeomorphism extends to a framing-preserving contactomorphism from a neighborhood of $K_1\#_{\alpha_1,\alpha_2}K_2$ to a neighborhood of $K_1\#_{\beta_1,\beta_2}K_2$ (similarly for $J_1\#_{\alpha_1,\alpha_2}J_2$ and $J_1\#_{\beta_1,\beta_2}J_2$). Under this contactomorphism, the contact framings and the Seifert framings are identified, and the result follows. □

Corollary 5.11. *The value of $\widetilde{tb}_{\Sigma_1\#\Sigma_2}(K_1\#K_2, J_1\#J_2)$ in the relative Legendrian connected sum is independent of the choice of arcs α_i, B_i, f.*

Lemma 5.12. $\widetilde{tb}(K_1\#K_2, J_1\#J_2) = \widetilde{tb}_{\Sigma_1}(K_1, J_1) + \widetilde{tb}_{\Sigma_2}(K_2, J_2)$.

Proof. By definition, $\widetilde{tb}(K_1\#K_2, J_1\#J_2) = tw_{K_1\#K_2}(\xi, Fr_\Sigma) - tw_{J_1\#J_2}(\xi, Fr_\Sigma)$ or $\big(tw_{K_1}(\xi_1, Fr_{\Sigma_1}) + tw_{K_2}(\xi_2, Fr_{\Sigma_2}) + 1\big) - \big(tw_{J_1}(\xi_1, Fr_{\Sigma_1}) + tw_{J_2}(\xi_2, Fr_{\Sigma_2}) + 1\big)$ and rearranging terms, we obtain $\widetilde{tb}_{\Sigma_1}(K_1, J_1) + \widetilde{tb}_{\Sigma_2}(K_2, J_2)$. □

Proposition 5.13. *Consider a homologous Legendrian knot pair (K_i, J_i) in a tight contact 3-manifold (M_i, ξ_i) such that $\widetilde{tb}(K_i, J_i) = \widetilde{Tb}(K_i, J_i), i = 1, 2$. Then in the relative Legendrian connected sum $\widetilde{Tb}(K_1\#K_2, J_1\#J_2) \geq \widetilde{Tb}(K_1, J_2) + \widetilde{Tb}(K_2, J_2)$.*

Proof. By the construction of the relative Legendrian connected sum (Remark 5.4) and by Lemma 5.12 above, $\widetilde{tb}(K_1\#K_2, J_1\#J_2) = \widetilde{Tb}(K_1, J_1) + \widetilde{Tb}(K_2, J_2)$, which implies that $\widetilde{Tb}(K_1\#K_2, J_1\#J_2) \geq \widetilde{Tb}(K_1, J_2) + \widetilde{Tb}(K_2, J_2)$. □

Remark 5.14. (Splitting a relative Legendrian connected sum) Consider a tight contact 3-manifold (M, ξ) with a Legendrian knot pair $(K_1\#K_2, J_1\#J_2)$ which is a relative Legendrian connected sum of (K_1, J_1, M_1, ξ_1) with (K_2, J_2, M_2, ξ_2). Then there exists an embedded splitting 2-sphere $S \subset (M, \xi)$ such that $S \cap \Sigma$ is the union of arcs α_1 and α_2. Take a 2-disc $D \subset S$ with $\partial D = \gamma' \cup \alpha_1 \cup \gamma'' \cup \alpha_2$ (Figure 2).

Let $M = M_1' \cup M_2'$ with $\partial M_i' = S$, where S is a 2-sphere with appropriate orientation, and complete each M_i' by a standard contact 3-ball B. Then $M_i' \cup B$ is diffeomorphic to the original manifold M_i used in the relative Legendrian connected

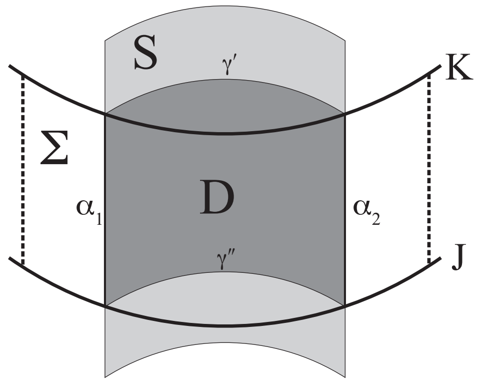

FIGURE 2. Splitting a relative Legendrian connected sum.

sum. Completing K by γ' and J by γ'' yields knots of type $K_i, J_i \subset M_i$. On the contact level, $\xi|_{M_i'}$ extends uniquely over B, and by Colin's theorem, each M_i' has a tight contact structure ξ_i. The arcs γ' and γ'' are Legendrian and contain one left-handed half-twist of the contact planes relative to Fr_S each. So the resulting knots are Legendrian and satisfy $\widetilde{tb}(K_1, J_2)+\widetilde{tb}(K_2, J_2) = \widetilde{tb}(K_1\#K_2, J_1\#J_2)$. The diffeomorphism type of the components that Σ is split into depends on the arcs α_i and surface S. However, knot types of the knots, the diffeomorphism type of the manifolds M_i, and the isotopy type of $\xi_i, i = 1, 2$ are all independent of α_i and S.

Proposition 5.15. *Let (M, ξ)be tight and let $(K_1\#K_2, J_1\#J_2)$ be a relative Legendrian connected sum pair with $\widetilde{tb}(K_1\#K_2, J_1\#J_2) = \widetilde{Tb}(K_1\#K_2, J_1\#J_2)$. Then $\widetilde{Tb}(K_1\#K_2, J_1\#J_2) \leq \widetilde{Tb}(K_1, J_2) + \widetilde{Tb}(K_2, J_2)$.*

Theorem 5.16. *In the relative Legendrian connected sum of homologous Legendrian knot pairs (K_i, J_i) in tight contact 3-manifolds, $\widetilde{Tb}(K_1\#K_2, J_1\#J_2) = \widetilde{Tb}(K_1, J_2) + \widetilde{Tb}(K_2, J_2)$.*

Remark 5.17. (Additivity of the relative rotation number) in some cases, from a global trivialization of the contact structure. The rotation number is measured as the difference of the upward and downward cusps, so $r(K_1\#K_2) = r(K_1)+r(K_2)$ in (S^3, ξ_{std}). By construction, the relative rotation numbers below are well-defined and independent of the choices made in the corresponding connected sum. In the relative Legendrian connected sum, the arcs α_1 and α_2 contribute the same amount to the rotation number. Since they get sent to one upward and one downward cusp under the (local) contactomorphism to (S^3, ξ_{std}), their total contribution to the rotation number is 0. Alternatively, note that the orientation-reversing diffeomorphism used in forming all of the connected sums identifies arcs along the summand knots with the same (and opposite in sign) contributions to the respective rotation numbers.

Proposition 5.18. *Given homologous Legendrian knots $K_1, J \subset (M_1, \xi_1)$ with $K_1 \cup J = \partial S_1$ and a Legendrian knot $K_2 \subset (M_2, \xi_2)$ with a Seifert surface $K_2 = \partial\Sigma_2$, assume that the ξ_i are tight. Then $\widetilde{r}_\Sigma(K_1\#K_2, J) = \widetilde{r}_{\Sigma_1}(K_1, J) + r_{\Sigma_2}(K_2)$, where J in the term $\widetilde{r}_\Sigma(K_1\#K_2, J)$ is a knot in $M_1\#M_2$.*

Theorem 5.19. *For a homologous Legendrian knot pair (K_i, J_i) in a tight contact 3-manifold (M_i, ξ_i) with Seifert surface $\Sigma_i, i = 1, 2$, assume (M_i, ξ_i) is tight.*

Then $\widetilde{r}_\Sigma(K_1\#K_2, J_1\cup J_2) = \widetilde{r}_{\Sigma_1}(K_1, J_1) + \widetilde{r}_{\Sigma_2}(K_2, J_2)$*, where* $J_1 \cup J_2$ *in the term* $\widetilde{r}_\Sigma(K_1\#K_2, J_1\cup J_2)$ *is a knot in* $M_1\#M_2$*.*

Remark 5.20. Combined with the classical additivity of the self-linking number under transverse connected sums, the constructions in this section (in particular, Remarks 5.2, 5.3, and 5.8, and Lemmas 5.4, 5.5) carry over to the transverse category to give the construction of relative transverse connected sums and additivity of the relative self-linking number under those. In particular, the generalized Bennequin inequality for a tight contact 3-manifold, implies that we would get an additivity of the maximal self-linking numbers of transverse knot pairs under relative transverse connect sums.

6. Relative Thurston-Bennequin invariant in $(S^1\times D^2, \xi_n)$

Consider $S^1\times D^2$ with $\xi_n = \ker\big(\sin(2\pi nz)dx + \cos(2\pi nz)dy\big), n \geq 1$, in local coordinates $\{z, \{x, y\}\}$, where ξ_n is generated by $\{\partial/\partial z,\ \cos(2\pi nz)\partial/\partial x - \sin(2\pi nz)\partial/\partial y\}$. The core curve $J = S^1\times\{(0,0)\}$ is Legendrian and, in fact, all curves of the form $\{(z,(x,y)) \mid x^2+y^2 < 1,\ z\in S^1\}$ are Legendrian. Intuitively, $S^1\times D^2$ is foliated by Legendrian curves parallel to the core J. The contact structure is invariant under translation in the plane $\{z = \textit{const.}\}$, however, it is not vertically invariant, in particular, it makes n left-handed 2π-twists along each (oriented) Legendrian curve parallel to the core J.

Since $S^1\times D^2\setminus J$ retracts to $\partial(S^1\times D^2)\cong S^1\times S^1$, $H_1(S^1\times D^2\setminus J)\cong\mathbb{Z}\oplus\mathbb{Z}$ is generated by oriented $\{\mu,\lambda\}$, where $\lambda = S^1\times\{p\},\ p\in\partial D^2$, and $\mu\subset(S^1\times D^2\setminus J)$ is the boundary of $\{0\}\times D^2$ with $\lambda\cdot\mu = 1$. Then for $K\subset(S^1\times D^2\setminus J)$, $[K] = n\mu + m\lambda\in H_1(S^1\times D^2\setminus J),\ n, m\in\mathbb{Z}$. Call n in $[K] = n\mu+m\lambda$ *the linking number* $lk(K, J)$ *of* K *with* J. Alternatively, this is the geometric intersection of K with the annulus $A = S^1\times[(0,0),(1,0)]$ so $lk(K,J) = K\cdot A$.

Consider a Legendrian knot K homologous to J in $(S^1\times D^2, \xi_n)$, $K\cup J = \partial\Sigma$. Denote the *Thurston-Bennequin invariant of* K *relative to* J by $\widetilde{tb}_{n,\Sigma}(K,J) := tw_K(\xi, Fr_\Sigma) - tw_J(\xi, Fr_\Sigma)$. It is well-defined (with $H_2(S^1\times D^2) = 0$ implying it is independent of the Seifert surface Σ) and depends on the integer n. So for a knot K homologous to the core J and $\partial\Sigma = K\cup J$ in $(S^1\times D^2, \xi_n)$, we will omit the subscript Σ, and use the notation $\widetilde{tb}_n(K,J) = tw_K(\xi, Fr_\Sigma) - tw_J(\xi, Fr_\Sigma)$. We want to study Legendrian isotopies of K across J.

Lemma 6.1. *Fix a number* $r\in(0,1)$ *and let* $[p, q$ *denote the line segment in* $D^2\subset\mathbb{R}^2$ *from point* p *to point* q*. There exists an annulus* $A = S^1\times[(0,0),(a,b)]$ *with* $a^2+b^2 < r$*, such that* $K\subset S^1\times\{(x,y) : x^2+y^2\leq a^2+b^2\}$ *and* $K\pitchfork A$*.*

Proof. Since K is properly embedded in $S^1\times D^2$, it is contained in a solid torus of the type $S^1\times\{(x,y)|x^2+y^2\leq r'\}$ for some $r'\in(0,1), r' < r$. Parametrize K by $t\longmapsto\{z(t), x(t), y(t)\}$, where $t\in S^1$ and consider the map $f : K\longrightarrow S^1$ given by $f : \{z(t), x(t), y(t)\}\longmapsto\theta\in S^1$ such that $x = \cos\theta$ and $y = \sin\theta$, in other words, θ is the angle that the segment $[(0,0),(x(t),y(t))]$ makes with the x-axis. So $f : S^1\to S^1$ is a smooth map and by Sard's theorem, almost every value of the map is a regular value, that is, the differential of f is onto everywhere. Then $f^{-1}(\theta)$ for a given regular value θ produces a set of transverse intersection points of K with the annulus $A = S^1\times\{(x,y) : x^2+y^2\leq r\}$. $\square$

Lemma 6.2. *The annulus A traces out a Legendrian isotopy from $J = S^1 \times \{(0,0)\}$ to $J' = S^1 \times \{(c,d)\}$, $c^2 + d^2 = r$, with $lk(J', K) = 0$ and $A \pitchfork K$. It extends to an ambient contact isotopy of $(S^1 \times D^2, \xi_n)$ fixing the boundary.*

Proof. The annulus A is foliated by Legendrian knots of type $S^1 \times \{(x,y)\}$, all parallel copies of the core $J = S^1 \times \{(0,0)\}$, so it traces out a Legendrian isotopy between $J = S^1 \times \{(0,0)\}$ and $J' = S^1 \times \{(c,d)\}$. Since $J' \subset S^1 \times \{(x,y) : x^2 + y^2 \leq r' < r\}$, J' and K are unlinked, J' co-bounds an annulus A' with $S^1 \times \{(1,0)\}$ and $K \cap A' = \emptyset$. This Legendrian isotopy extends to an ambient contact isotopy of $(S^1 \times D^2, \xi_n)$ and can be arranged to be the identity on the boundary. □

Lemma 6.3. *The inverse of the isotopy traced out by A is an ambient contact isotopy sending J' to J such that the image of K is a Legendrian knot that crosses J transversely to become unlinked from J.*

Lemma 6.3 says that for homologous K and J, Legendrian isotoping K across J does not change the value of $\widetilde{tb}_\Sigma(K, J)$. From the construction, such a Legendrian isotopy always exists.

Proposition 6.4. *Given a Legendrian knot K homologous to $J = S^1 \times \{(0,0)\}$ in $(S^1 \times D^2, \xi_n)$, there exists an ambient contact isotopy, identity on the boundary, from K to K' such that $lk(K', J) = 0$ and $\widetilde{tb}_n(K', J) = \widetilde{tb}_n(K, J)$.*

Lemma 6.5. *After an unlinking Legendrian isotopy of K as in Proposition 6.4, $tw_J(\xi_n, Fr_{\Sigma'}) = n$ and $\widetilde{tb}_n(K) = tw_K(\xi_n, Fr_{\Sigma'}) - n$, where $K' \cup J = \partial\Sigma'$.*

Proof. Once an unlinking Legendrian isotopy is applied to K, the Seifert surface Σ' for $K' \cup J$ induces a Seifert framing $Fr_{\Sigma'}$ on J which is equal to the product framing on J (a push-off J' of J into Σ' which defines the framing must vanish in the first homology of the complement). Therefore, $tw_J(\xi_n, Fr_{\Sigma'}) = n$. □

Let κ be a smooth knot type in $S^1 \times D^2$, and let $\mathcal{L}_n(\kappa)$ denote *the set of Legendrian representatives of κ homologous to the core J in* $(S^1 \times D^2, \xi_n)$.

Lemma 6.6. *The function $\widetilde{tb}_n : \mathcal{L}_n(\kappa) \longrightarrow \mathbb{Z}$ is not bounded below.*

Proof. Take any Legendrian representative $K \in \kappa$ and stabilize K. The resulting knot $K' \in \kappa$ has $\widetilde{tb}_n(K') = \widetilde{tb}_n(K) - 1$ with Seifert surface Σ' obtained from Σ by adding a half-disc (and smoothing corners). Σ for K and Σ' are smoothly isotopic, and their boundaries are smoothly but not Legendrian isotopic,with $K' \in \kappa$. Repeating this lowers the relative Thurston-Bennequin invariant arbitrarily. □

Theorem 6.7. *The function $\widetilde{tb}_n : \mathcal{L}_n(\kappa) \to \mathbb{Z}$ is bounded above when $n < 0$.*

Proof. Apply an unlinking Legendrian isotopy to a Legendrian representative K of κ in $(S^1 \times D^2, \xi_n)$ homologous to the core $J = S^1 \times \{0\}$. Consider a Legendrian unknot K_0 in (S^3, ξ_{std}) and a 2-disc D bounded by K_0. Arrange that $tw_{K_0}(\xi_{std}, Fr_D) = tb(K_0) = n < 0$. Since $Tb = -1$ for the trivial knot type in (S^3, ξ_{std}), stabilization allows us to construct such a Legendrian unknot. Now take a framed Legendrian neighborhood $(N(K_0), \xi_{std}|_{N(K_0)}) \simeq (S^1 \times D^2, \xi_n)$, where K_0 is sent to $J = S^1 \times \{0,0\}$, the Seifert framing on K_0 given by the product framing on $S^1 \times \{0,0\}$. We have a framing-preserving contactomorphism $\varphi : (S^1 \times D^2, \xi_n) \to (N(K_0), \xi_{std}|_{N(K_0)})$. Then $\varphi(K) \subset N(K_0)$ is unlinked from K_0 and cobounds a surface $\varphi(\Sigma)$ with K_0 such that $tw_{K_0}(\xi_{std}, Fr_{\varphi(\Sigma)}) = n$. Since both framings Fr_D

and $Fr_{\varphi(\Sigma)} = \varphi_*(Fr_\Sigma)$ on K_0 are Seifert, they are both given by push-offs into each respective surface which vanish in $H_1(N(K_0) \setminus K_0)$. Thus, we can isotop the interiors of D and $\varphi(\Sigma)$ so that in a neighborhood of K_0 they intersect only in K_0. Away from that neighborhood, however, the interior of D may intersect the interior of $\varphi(\Sigma)$ and/or the knot $\varphi(K) \subset N(K_0)$. The possible intersections are arcs and closed curves, which are eliminated standardly (see [17]) by locally isotoping the interior of D without changing $tb_D(K_0)$ or $tb_{\varphi(\Sigma)}\varphi(K)$. Now $\varphi(\Sigma)$ and D intersect only in K_0. So $\Sigma' = \varphi(\Sigma) \cup D$ is a Seifert surface for $\varphi(K)$. Since φ is framing-preserving, $tw_K(\xi_n, Fr_\Sigma) = tw_{\varphi(K)}(\varphi_*(\xi_n), Fr_{\Sigma'}) = tw_{\varphi(K)}(\xi_{std}|_{N(K_0)}, Fr_{\Sigma'}) = tb_{\Sigma'}(\varphi(K))$. Therefore, $\widetilde{tb}_n(K) = tw_K(\xi_n, Fr_\Sigma) - n = tb_{\Sigma'}(\varphi(K)) - n$ and $tb_{\Sigma'}(\varphi(K))$ is bounded above by the maximal Thurston-Bennequin invariant for the knot type of $\varphi(K)$ in S^3. This upper bound is independent of φ and only depends on the smooth knot type of K in $S^1 \times D^2$. Therefore, $\widetilde{tb}_n(K) = tb_{\Sigma'}(\varphi(K)) - n \leq Tb(\varphi(K)) - n$ so $\widetilde{tb}_n(K)$ is bounded above. $\square$

For $K \subset (S^1 \times D^2, \xi_n)$, let $\widetilde{Tb}_n(K) = \max\{\widetilde{tb}_n(K) \mid K \text{ is of type } \kappa\}$.

7. Limitations of the relative Legendrian knot invariants

Let $K \subset (S^3, \xi_{std})$ be a null-homologous Legendrian knot, and let $J' = S^1 \times \{0, \frac{1}{2}\}$ in $(S^1 \times D^2, \xi_n)$ with $J \cup J' = \partial A$ for $A = S^1 \times \{(x, y) | x = 0, 0 \leq y \leq \frac{1}{2}\}$. Then $\widetilde{tb}(J', J) = 0$. Form the Legendrian connected sum $K' = J' \# K$ in $(S^1 \times D^2, \xi_n) \# (S^3, \xi_{std}) \cong (S^1 \times D^2, \xi_n)$ with Seifert surface Σ'. Then $\widetilde{tb}_{\Sigma'}(K', J) = \widetilde{tb}_A(J', J) + tb_\Sigma(K) + 1 = tb_\Sigma(K) + 1$ and $\widetilde{r}_{\Sigma'}(K', J) = \widetilde{r}_A(J', J) + r_\Sigma(K) = r_\Sigma(K)$.

Consider Chekanov's examples of Legendrian embeddings $K_1, K_2 \subset (S^3, \xi_{std})$ of the 5_2 knot with $tb(K_1) = tb(K_2)$ and $r(K_1) = r(K_2)$, yet not Legendrian isotopic (see [3]). The knots $K_1' = J' \# K_1$ and $K_2' = J' \# K_2$ in $(S^1 \times D^2, \xi_n)$ are homologous to the core J with $\widetilde{tb}_n(K_i', J) = tb(K_i) + 1$ and $\widetilde{r}_n(K_i', J) = r(K_i)$.

Lemma 7.1. *The knots $K_1', K_2' \subset (S^1 \times D^2, \xi_n)$ are not Legendrian isotopic.*

Proof. If K_1' and K_2' were Legendrian isotopic, then we embed $(S^1 \times D^2, \xi_n)$ in (S^3, ξ_{std}) as a framed Legendrian neighborhood of an unknot U with $tb = -n$ (see Section 4). The knots K_i' are unlinked from J so $\varphi(J)$ bounds a 2-disc D disjoint from the Seifert surface of each $\varphi(K_i')$. This produces two Legendrian isotopic knots with $tb(\varphi(K_i')) = \widetilde{tb}_n(K_i', J) - n = tb(K_i) - n + 1$ and $r(\varphi(K_i')) = \widetilde{r}_n(K_i') = r(K_i) + r_D(U)$. Since $tb(K_1) = tb(K_2)$ and $r(K_1) = r(K_2)$, we have two Legendrian embeddings that are both stabilizations of of the Legendrian representatives K_i of 5_2 knot in (S^3, ξ_{std}) with equal invariants and are Legendrian isotopic, contradicting Chekanov's result. $\square$

This strongly suggests that the relative invariants exhibit the same limitations as their classical analogues. Arguing this in general would follow a similar argument.

8. Legendrian knots which cobound an embedded annulus

We will prove the following general theorem. The special case for $(S^1 \times D_2, \xi_n)$ with $J \subset (S^1 \times D_2, \xi_n)$ denoting the Legendrian core follows directly.

Theorem 8.1. *Let K, J be Legendrian knots in a tight contact 3-manifold (M, ξ) cobounding an embedded annulus $A \hookrightarrow (M, \xi)$ with $\widetilde{tb}_A(K, J) = 0$ and $\widetilde{r}_A(K, J) = 0$. There is a global contact isotopy of (M, ξ) (fixing ∂M if $\partial M \neq \emptyset$) sending K to J.*

Lemma 8.2. *Given two Legendrian knots $K, J \subset (M, \xi)$ which cobound an annulus A with $\widetilde{tb}_A(K, J) = 0$, then $tw_K(\xi, Fr_A) = tw_J(\xi, Fr_A) \leq 0$.*

Proof. By Lemma 4.4, $tw_K(\xi, Fr_A) + tw_J(\xi, Fr_A) + |r_A(K) + r_A(J)| \leq -\chi(A) = 0$ which implies $tw_K(\xi, Fr_A) = tw_J(\xi, Fr_A) \leq 0$. $\square$

The above lemma and Honda's theorem ([19]), which extends Giroux's results to surfaces with boundary, imply that A can be isotoped to be convex, rel ∂A, C^0-small near ∂A and C^∞-small away from ∂A. When $tw_K(\xi, Fr_A) = tw_J(\xi, Fr_A) < 0$, we prove Theorem 8.1 by foliating A by Legendrian knots parallel to the boundary thus tracing out a Legendrian isotopy between K and J.

Remark 8.3. The argument do not apply when $tw_K(\xi, Fr_A) = tw_J(\xi, Fr_A) = 0$. In this case, we stabilize the Legendrian knots and then apply this argument.

Lemma 8.4. *Given two Legendrian knots $K, J \subset (M, \xi)$ which cobound an annulus A with $\widetilde{tb}_A(K, J) = \widetilde{r}_A(K, J) = 0$ and $tw_K(\xi, Fr_A) = tw_J(\xi, Fr_A) < 0$, A can be isotoped to be convex rel boundary with characteristic foliation with Legendrian leaves parallel to the boundary components.*

First Proof of Lemma 8.4. Note $tw_K(\xi, Fr_A) = tw_J(\xi, Fr_A) < 0$ implies that the dividing set Γ_A has a nonempty intersection with K and J. Take another convex annulus A' with $\partial A' = K \cup J$ such that $A \cap A' = K \cup J$. Edge-Round along K and J to build a convex torus $T = A \cup A'$. This process uses standard framed Legendrian neighborhoods around J and K and replaces their intersection with $A \cup A'$ by a smooth embedded surface. Locally, this is a Legendrian isotopy of K and J to knots K', J' (see [19]). By tightness and Giroux's Criterion, Γ_T consists of an even number of parallel dividing curves that are not meridional. Note that $K, J \subset T$ are of the same homology class. Also, $tw_K(\xi, Fr_A) = tw_J(\xi, Fr_A) < 0$ implies that $tw_{K'}(\xi, Fr_T) = tw_{J'}(\xi, Fr_T) < 0$, so J' and K' intersect Γ_T. We can isotop T to be foliated by leaves parallel to the knots K' and J' using Giroux's Flexibility theorem ([13]). This is a Legendrian isotopy of K' and J', sending them to K'' and J'' on the new convex torus T' where they are Legendrian isotopic through the leaves. By the Legendrian Isotopy Extension theorem, the composition of these isotopies is a global contact isotopy. Note that we used $\widetilde{r}_A(K, J) = 0$ to build T. $\square$

Second Proof of Lemma 8.4. Parametrize A, K, J as $A = \varphi(S^1 \times [-\frac{1}{2}, \frac{1}{2}]), J = \varphi(S^1 \times \{-\frac{1}{2}\}), K = \varphi(S^1 \times \{\frac{1}{2}\})$ in M. Extend φ to an embedding $S^1 \times D^2 \hookrightarrow (M, \xi)$. Consider a closed solid torus neighborhood T of A. Consider a diffeomorphism $f : T \to S^1 \times D^2$ such that $f : K \mapsto S^1 \times \{p_1\}$ and $f : J \mapsto S^1 \times \{p_2\}$ for $p_1,\ p_2 \in D^2$. Let $S^1 \times D^2$ be equipped with the contact structure ξ_n, where $tw_J(\xi, Fr_A) = tw_K(\xi, Fr_A) = -n$. Note $f(J)$ and $f(K)$ are Legendrian in $(S^1 \times D^2, \xi_n)$, but they are also Legendrian in $(S^1 \times D^2, f_*(\xi|_T))$. So consider the Legendrian isotopy between them given by just sending $S^1 \times \{p_1\}$ to $S^1 \times \{p_2\}$ through parallel copies $g_t : S^1 \times (1-t)p_1 + tp_2$. This is a Legendrian isotopy inside $(S^1 \times D^2, \xi_n)$. Since f is a diffeomorphism rel boundary, it is a contactomorphism, and by the uniqueness of the tight contact structure ξ_n, there is an isotopy sending ξ_n to $f_*(\xi|_T)$. The inverse of this isotopy composed with the Legendrian isotopy from $f(K)$ to $f(J)$ and the inverse of f yields a Legendrian isotopy from J to K in T. $\square$

Third Proof of Lemma 8.4. Note that Γ_A has a component running from K to J. To see this, assume Γ_A consists only of boundary-parallel dividing arcs. Then in the

construction of the convex torus $T = A \cup A'$ above, the dividing set would contain a trivial closed curve, contradicting Giroux's Criterion. Therefore, the annulus A necessarily has a boundary-to-boundary dividing arc (an even number of these). $\square$

Remark 8.5. Co-bounding an embedded annulus is a transitive relation of knots. In particular, the knots in this relation are smoothly isotopic.

Lemma 8.6. *Let K_i be a framed knot with framing Fr_i, $i = 1, 2, 3$ with $\partial A_i = K_i \cup K_{i+1}$, $i = 1, 2$ for embedded annuli A_i. There exists an embedded annulus A with $\partial A = K_1 \cup K_3$ such that $tw_{K_1}(Fr_A, Fr_1) = tw_{K_1}(Fr_{A_1}, Fr_1) + K_1 \cdot A_2$ and, similarly, $tw_{K_3}(Fr_A, Fr_3) = tw_{K_3}(Fr_{A_2}, Fr_3) + K_3 \cdot A_1$.*

Proof. Resolve the intersections of A_1 and A_2 to get an embedded annulus A. $\square$

Lemma 8.7. *Let K_1 and K_2 be Legendrian knots which cobound annuli A_i, respectively, with a Legendrian knot J in a tight contact 3-manifold (M, ξ). If $\widetilde{tb}_{A_1}(K_1, J) = \widetilde{tb}_{A_2}(K_2, J)$ and $\widetilde{r}_{A_1}(K_1, J) = \widetilde{r}_{A_2}(K_2, J)$, then K_1 and K_2 are Legendrian isotopic.*

Proof. Apply Lemma 8.4 together with Lemma 8.6. $\square$

9. Legendrian knots isotopic to the core in $(S^1 \times D^2, \xi_n)$.

Let K be isotopic to J. The generator of ker $\big(H_1(S^1 \times D^2 \setminus K) \to H_1(S^1 \times D^2)\big)$ is a curve μ so that the 0-framing of K in $S^1 \times D^2$ is defined by K' with $[K'] = 0 \cdot \mu$, where $[K']$ is unique up to a choice for a generator of the other factor in $H_1(\partial(S^1 \times D^2 \setminus K)) \cong \mathbb{Z} \oplus \mathbb{Z}$. A Legendrian K has a twisting number $tb_n(K)$ defined as $lk(K, K')$ for a push-off K' in the normal direction to the contact planes along K, so $tb_n(K) = lk(K, K') = m$, where m is the unique integer with $[K'] = m \cdot \mu$. Now embed $(S^1 \times D^2, \xi_n)$ in (S^3, ξ_{std}) as the framed neighborhood of a Legendrian unknot U with $tb(U) = -n = tb_n(J)$. Then K is a Legendrian unknot in (S^3, ξ_{std}) smoothly isotopic to U. Note that $tb_n(K) = tb_{D_K}(K)$, where $\partial D_K = K$ in S^3. Similarly, the global trivialization of ξ_n in $S^1 \times D^2$ given by $\partial/\partial z$ gives the rotation number $r_n(K)$. After embedding $(S^1 \times D^2, \xi_n)$ standardly into (S^3, ξ_{std}), we have $r_{D_K}(K) = r_n(K)$.

With this in mind, we classify Legendrian knots smoothly isotopic to the Legendrian core in $(S^1 \times D^2, \xi_n)$ with equal tb_n and r_n.

Lemma 9.1. *Let K be isotopic to $J = S^1 \times \{(0, 0)\}$ in $(S^1 \times D^2, \xi_n)$, $K \cap J = \emptyset$. Then there exists an embedded annulus $A \hookrightarrow S^1 \times D^2$ with $\partial A = K \cup J$.*

Proof. Embed $(S^1 \times D^2, \xi_n)$ in (S^3, ξ_{std}) as the framed neighborhood of a Legendrian unknot U with $tb(U) = tb_n(J) = -n$, The images of K and J are isotopic in S^3, and so are the discs D_K and D_J that they bound. In particular, we can isotop them outside the interior of $S^1 \times D^2$ so that D_K and D_J coincide in there and in a curve $\gamma \subset \partial(S^1 \times D^2)$. Consider the annuli $A_K = D_K \cap (S^1 \times D^2)$ and $A_J = D_J \cap (S^1 \times D^2)$ and resolve their intersections away from γ as in [17] to obtain an embedded annulus A with $\partial A = K \cup J$ and the framings along K and J change by the same number $lk_{A_J}(K, J) = K \cdot A_J = lk_{A_K}(J, K) = J \cdot A_K$. $\square$

Lemma 9.2. *For $A \hookrightarrow S^1 \times D^2$ above, we have $\widetilde{tb}_A(K, J) = 0$ and $\widetilde{r}_A(K, J) = 0$.*

Proof. For Legendrian K and J, we have $tb_A(K) = tb_A(J)$ since $tb_n(K) = tb_n(J)$ so $\widetilde{tb}_A(K, J) = tb_A(K) - tb_A(J) = (tb_n(K) + lk_{A_J}(K, J)) - (tb_n(J) + lk_{A_K}(J, K)) = 0$.

Similarly, $\widetilde{r}_A(K,J) = r_n(K) - r_n(J) = r_A(K) - r_A(J) = 0$. Therefore A traces out a Legendrian isotopy between K and J, by Theorem 8.4. $\square$

Remark 9.3. Recall we assumed in [17] that the Legendrian isotopy crossing the reference knot J was locally embedded. Lemma 9.2 shows this assumption is justified. The converse is not generally true, and finding an embedded annulus which traces out a Legendrian (or even smooth) isotopy is not generally possible. It is a good problem to find the obstructions for an isotopy to be embedded.

10. Further classification results

Let $K, J_1, \ldots, J_m$ be Legendrian knots in a contact 3-manifold (M, ξ) with Seifert surface Σ. The relative invariants $\widetilde{tb}_\Sigma(K, J_1 \cup \cdots \cup J_m) = tw_K(\xi, Fr_\Sigma) - \sum_{k=1}^m tw_{J_k}(\xi, Fr_\Sigma)$ and $\widetilde{r}_\Sigma(K, J_1 \cup \cdots \cup J_m) = \omega(v_K) - \sum_{k=1}^m \omega(v_{J_i})$ are well-defined ([17]), in particular, they are invariant under Legendrian isotopy of K which fixes the J_i ([17]).

Lemma 10.1. *Let $J_i = S^1 \times \{p_i\}, p_i \in D^2$ be m parallel copies of $J = S^1 \times \{(0,0)\}$ in $(S^1 \times D^2, \xi_n)$ and let K be a Legendrian knot with $K \cup J_1 \cup \cdots \cup J_m = \partial\Sigma$. The J_i may be Legendrian isotoped so that $tw_{J_i}(\xi_n, Fr_\Sigma) = tw_J(\xi_n, Fr_{S^1\times D^2}) = tb_n(J) = -n$ and $\widetilde{r}_\Sigma(K, J_1 \cup \cdots \cup J_m) = r_n(K) - \sum_{i=1}^m r_n(J_i)$.*

Proof. The method of unlinking Legendrian isotopy (Proposition 6.4) allows us to unlink J_i from K so that the Seifert framing and the product framing coincide, without changing the relative invariants (for the invariance of the relative rotation number, note that the contact structure is globally trivial). Then $tw_{J_i}(\xi_n, Fr_\Sigma) = tw_J(\xi_n, Fr_{S^1\times D^2}) = -n$ so $\widetilde{tb}_\Sigma(K, J_1 \cup \cdots \cup J_m) = tw_K(\xi_n, Fr_\Sigma) - mn$ and $\widetilde{r}_\Sigma(K, J_1 \cup \cdots \cup J_m) = \omega(v_K) - \sum_{i=1}^m \omega(v_{J_i}) = \omega(v_K) - m\omega(v_J)$. $\square$

Remark 10.2. Let K be a Legendrian knot which cobounds an m-punctured 2-disc D with a collection of m Legendrian copies of J in $(S^1 \times D^2, \xi_n)$. Then if m is odd, K is smoothly isotopic to J and if m is even then K is homotopically trivial.

Lemma 10.3. *Let K_1 and K_2 be two Legendrian knots each cobounding an m-punctured 2-disc D_i with m copies of J in $(S^1 \times D^2, \xi_n)$. Assume $\widetilde{tb}_{D_1}(K_1, J_1' \cup \cdots \cup J_m') = \widetilde{tb}_{D_2}(K_2, J_1'' \cup \cdots \cup J_m'')$ and $\widetilde{r}_{D_1}(K_1, J_1' \cup \cdots \cup J_m') = \widetilde{r}_{D_2}(K_2, J_1'' \cup \cdots \cup J_m'')$. Then if m is odd, there exists an embedded annulus A with $\partial A = K_1 \cup K_2$ such that $\widetilde{tb}_A(K_1, K_2) = 0$ and $\widetilde{r}_A(K_1, K_2) = 0$. If m is even, there exists an embedded 2-dsic D_i with $K_i = \partial D_i$ such that $tb_{D_1}(K_1) = tb_{D_2}(K_2)$ and $r_{D_1}(K_1) = r_{D_2}(K_2)$.*

Proof. We use Proposition 6.4 to isotop all J_i' and J_j'' to a neighborhood $N(\widetilde{J})$ of a copy of J with the $K_i \subset (S^1 \times D^2) \setminus N(\widetilde{J})$. The relative invariants are fixed and $tw_{J_i'}(\xi_n, Fr_{D_1}) = tw_{J_j''}(\xi_n, Fr_{D_2}) = tb_n(J) = -n$ and $\widetilde{r}_{D_1}(J_i') = r_n(J_i') = \widetilde{r}_{D_2}(J_j'')$. Now Legendrian isotop each J_i' to a J_j'' through parallel copies of J. This may create circle intersections between D_1 and D_2, but there is an arrangement of J_i' getting mapped to the J_j'' which avoids all circle intersections. Also, $tw_{J_i'}(\xi_n, Fr_{D_1}) = -n = tw_{J_j''}(\xi_n, Fr_{D_2})$ implies that $tw_{K_1}(\xi_n, Fr_{D_1}) = tw_{K_2}(\xi_n, Fr_{D_2})$ and similarly $r_n(K_1) = \omega(v_{K_1}) = r_n(K_2) = \omega(v_{K_2})$. After all circles are eliminated, $\Sigma = D_1 \cup D_2$ has no self-intersections near the $J_i' = J_i''$ and arc self-intersections α_k running only from K_1 to K_2 and possibly some other circle intersections. If m is odd, then K_1 and K_2 are smoothly isotopic to the core in $S^1 \times D^2$ and are thus smoothly

isotopic, with the union of the K_i and α_k cobounds a collection of disjoint 2-discs in $S^1 \times D^2$, whose union is an annulus A with $K_1 \cup K_2 = \partial A$. If m is even, then K_1 and K_2 are (trivial and therefore) isotopic in $S^1 \times D^2$, and the above annulus traces out the Legendrian isotopy between them. □

Theorem 10.4. *Let K_1 and K_2 be two smoothly isotopic Legendrian knots each cobounding an m-punctured 2-disc D_i with m copies of J in $(S^1 \times D^2, \xi_n)$. If $\widetilde{tb}_{D_1}(K_1, J'_1 \cup \cdots \cup J'_m) = \widetilde{tb}_{D_2}(K_2, J''_1 \cup \cdots \cup J''_m)$ and $\widetilde{r}_{D_1}(K_1, J'_1 \cup \cdots \cup J'_m) = \widetilde{r}_{D_2}(K_2, J''_1 \cup \cdots \cup J''_m)$, then K_1 and K_2 are Legendrian isotopic.*

This can be translated to links in (S^3, ξ_{std}) by considering J as a trivial link component. Consider two smoothly isotopic $(m+1)-$*component* Legendrian links $L_1 = K_1 \cup J_1 \cup J_2 \cup \cdots \cup J_m$ and $L_2 = K_2 \cup J'_1 \cup J'_2 \cup \cdots \cup J'_m$ in (S^3, ξ_{std}) consisting only of trivial components and with Seifert surfaces D_1, D_2 which are m-punctured 2-discs such that K_i bounds a 2-disc whose interior is disjoint from D_i for $i = 1, 2$. Now take a Legendrian unknot U_i that links with L_i so that $lk_{D_1}(U_1, J_k) = lk_{D_2}(U_2, J'_k) = \pm 1$ so the new links $L_i \cup U_i$ are smoothly isotopic. Then after appropriate stabilizations of some or all of their components, the links $L_1 \cup U_1$ and $L_2 \cup U_2$ are Legendrian isotopic. To see this, take a framed Legendrian neighborhood of U_i, such that the J_i are meridians, isotop the U_i to coincide, and extend to a global contact isotopy. Then the complement of the now single solid torus is a solid torus to which the above theorem applies.

11. Connected sums and Legendrian simple knots.

Etnyre and Honda [11] connected sums of Legendrian simple knots extensively. We are interested in the relative version of their results. Let $\mathfrak{L}$ be the set of knot types whose Legendrian embeddings in (M_1, ξ_1) are Legendrian simple (i.e., classified by their classical invariants). Let $\mathfrak{L}'$ be the set of knot types $K_1 \# J$, where $K_1 \in \mathfrak{L}$ and J is a Legendrian knot in a tight contact 3-manifold (M_2, ξ_2). Consider an embedded annulus A in M_2 bounded by J and another Legendrian knot K_2, so $K_1 \# K_2$ in $(M_1, \xi_1) \# (M_2, \xi_2)$ is homologous to J. For a Legendrian representative K of $K_1 \# K_2$ in (M, ξ), the relative invariants $\widetilde{tb}(K, J)$ and $\widetilde{r}(K, J)$ are well-defined (??? and Theorem 5.1). Let $\mathfrak{L}'' \subset \mathfrak{L}'$ denote knot types in $\mathfrak{L}'$ whose Legendrian representatives in (M, ξ) are classified by the relative invariants with respect to J (call them *relatively Legendrian simple with respect to* J).

Question 11.1. When does the relative connected sum give a bijection $\mathfrak{L} \iff \mathfrak{L}''$? What are the obstructions, and when is the map only injective or surjective?

Etnyre-Honda [11] constructed connected sums of Legendrian simple knots in S^3 which are not Legendrian simple so $\mathfrak{L} \to \mathfrak{L}''$ is not always a bijection.

Lemma 11.2. *There is a bijective correspondence between $\mathfrak{L}$ and $\mathfrak{L}''$ in the case when $(M_i, \xi_i) = (S^3, \xi_{std})$, $i = 1, 2$ and $J \subset (M_2, \xi_2)$ is the unknot.*

This lemma is fairly straightforward to prove. We prove a generalization below.

Lemma 11.3. *There is a bijective correspondence between $\mathfrak{L}$ and $\mathfrak{L}''$ for $(M_1, \xi_1) = (S^3, \xi_{std})$, $(M_2, \xi_2) = (S^1 \times D^2, \xi_n)$, and $J \subset (S^1 \times D^2, \xi_n)$ is the Legendrian core.*

Proof. The mapping is given by the relative Legendrian connected sum. To see that it is onto, we take a knot type $\kappa_{S^1 \times D^2} \in \mathfrak{L}''$ and note that the splitting of

a Legendrian connected sum gives a unique knot type κ_{S^3}, so we need to show $\kappa_{S^3} \in \mathfrak{L}$. Given Legendrian knots K_i' in (S^3, ξ_{std}) with $K_i' = \partial\Sigma_i'$ of smooth knot type κ_{S^3} with equal classical invariants $tb_{\Sigma_i'}(K_i')$ and $r_{\Sigma_i'}(K)$, form the Legendrian connected sums $K_i = K_i' \# J_i' \subset (S^1 \times D^2, \xi_n) \cong (S^3, \xi_{std}) \# (S^1 \times D^2, \xi_n)$. Note that J_i' has equal relative invariants with respect to J, and $K_i' = K \# J'$ is homologous to J via the Seifert surface $\Sigma' = \Sigma \# A$. Therefore, $\widetilde{tb}_{\Sigma_i'}(K_i', J) = tb_{\Sigma_i}(K_i) + 1$ and $\widetilde{r}_{\Sigma_i'}(K_i', J_i') = r_{\Sigma_i}(K_i)$. So the K_i' are relatively Legendrian isotopic in $(S^1 \times D^2, \xi_n)$. Extend this Legendrian isotopy to a contact isotopy of $(S^1 \times D^2, \xi_n)$ such that the separating 2-spheres and the 3-balls they bound in $(S^1 \times D^2, \xi_n)$ coincide. Thus, the Legendrian isotopy reduces to an isotopy within the (now single) 3-ball containing the parts of the K_i' corresponding to the summands K_i. Now consider a contact isotopy of (S^3, ξ_{std}) which Legendrian isotops the knots K_i' so that they coincide at a point (on a neighborhood of that point, in fact). Such an isotopy exists (in the front projection, it is a horizontal and vertical translation). So then we can use the Legendrian isotopy fixing a common point that we found in the 3-ball above. The composition of these two Legendrian isotopies gives us the desired Legendrian isotopy from K_1 onto K_2 in (S^3, ξ_{std}). Note that we are using the classification of tight contact structures in S^3 and the standard 3-ball. To see that the mapping is injective, note that the relative Legendrian connect sum defines the knot type $\kappa_{S^1 \times D^2}$ uniquely for a given knot type κ_{S^3} in the smooth category. For a Legendrian simple κ_{S^3}, we show that $\kappa_{S^1 \times D^2}$ is relatively Legendrian simple. Consider two Legendrian knots K_1 and K_2 in $(S^1 \times D^2, \xi_n)$ of knot type$\kappa_{S^1 \times D^2}$ with equal relative invariants. Then $K_i = K_i' \# J_i$ for K_1' and K_2' of knot type κ_{S^3}. The relative Legendrian connected sum replaces a standard 3-ball neighborhood of a point $p_i \in J_i'$ with the standard 3-ball complement of a 3-ball neighborhood of a point on each knot K_i'. Use a contact isotopy to Legendrian isotop J_1' to J_2' so that the points p_i coincide and the 3-ball neighborhoods of those coincide as well. After the connected sum, we further extend the isotopy using the Legendrian isotopy between the K_i' from a 3-ball complement of a point in S^3 (this is contained in the standard 3-ball that they are embedded in). Thus, $K_i = K_i' \# J_i'$ are Legendrian isotopic in $S^1 \times D^2$ provided that the relative classical invariants of the knots $K_i = K_i' \# J_i'$ with respect to J are equal. $\square$

Remark 11.4. In order to piece together the Legendrian isotopies, equality of the relative invariants is not sufficient, we may need to distribute stabilizations among the components. Thus the bijective correspondence holds up to an equivalence relation that accounts for this (see Theorem 12.1 and compare with Theorem 3.4 in [11]).

Remark 11.5. The results in Section 7 follow from Lemma 11.3. Moreover, the argument generalizes to classify all Legendrian simple knot types in (S^3, ξ_{std}) as relatively Legendrian simple knot types in $(S^1 \times D^2, \xi_n)$.

Lemma 11.3 applies to Legendrian links in (S^3, ξ_{std}) with a trivial component.

Theorem 11.6. *Let $K \in \mathfrak{L}$ and U be an unknot. Legendrian simple links $K \cup U$ in (S^3, ξ_{std}) are classified by the link type and relative invariants of K relative to U.*

12. The relative structure theorem

A homologous knot pair (K, J) in a tight contact 3-manifold (M, ξ) is *relatively prime* if $(K, J) = (K_1, J_1)\#(K_2, J_2)$ in $(M_1, \xi_1)\#(M_2, \xi_2)$ implies $K_1 \subset M_1$ or $K_2 \subset M_2$ is the unknot. Let $S_\pm(K)$ be the positive/negative stabilization of K and $\mathfrak{L}_{(\kappa, M, \xi)}$ be the set of isotopy classes of Legendrian representatives of κ in (M, ξ). We have a relative version of Theorem 3.4 by Etnyre-Honda in [11].

Theorem 12.1. *Let $(\kappa, \gamma) = (\kappa_1, \gamma_1)\# \cdots \#(\kappa_n, \gamma_n)$ be a relative connected sum decomposition in a tight (M, ξ) into relatively prime pairs $(\kappa_i, \gamma_i) \subset (M_i, \xi_i)$. The map $\left(\dfrac{\left(\mathfrak{L}_{\kappa_1}, \mathfrak{L}_{\gamma_1}\right) \times \cdots \times \left(\mathfrak{L}_{\kappa_n}, \mathfrak{L}_{\gamma_n}\right)}{\sim}\right) \longrightarrow \left(\mathfrak{L}_{\kappa_1 \# \cdots \# \kappa_n}, \mathfrak{L}_{\gamma_1 \# \cdots \# \gamma_n}\right)$ is a bijection, where $\sim$ is defined by*

(1) $\Big(\ldots, (S_\pm(K_i), J_i), \ldots, (K_j, S_\pm(J_j)), \ldots\Big) \sim$

$$\sim \Big(\ldots, (K_i, S_\pm(J_i)), \ldots, (S_\pm(K_j), J_j), \ldots\Big)$$

(2) $\Big((K_1, J_1), \ldots, (K_n, J_n)\Big) \sim \sigma\Big((K_1, J_1), \ldots, (K_n, J_n)\Big)$*, where σ is a permutation of (κ_i, γ_i) so that $\sigma(M_i, \xi_i)$ is isotopic to (M_i, ξ_i) and $\sigma(\kappa_i, \gamma_i) = (\kappa_i, \gamma_i)$ for all i.*

This result follows directly from the structure theorem in [11], together with the construction of the relative Legendrian connected sum and the well-definedness of the relative invariants in [17].

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